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EFFECT OF DIFFERENT FACTORS ON THE ACCURACY OF THE SOLUTION OF A PARAMETRIZED INVERSE PROBLEM OF HEAT CONDUCTION

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Results are presented for a mathematical simulation of the effect of the error in approximation of the estimated function, the error in temperature measurements, and the error in specifying measurements on the accuracy of the solution of the parametrized boundary-value inverse problem.

Methods based on solving boundary-value inverse problems of heat conduction are widely used at present in the experimental investigation of processes of heat interaction of a solid body with the surrounding medium. In these problems we seek thermal boundary conditions and restore the temperature field in the body based on results of thermal measurements at separate internal points.

In many cases heat transfer in systems being investigated can be described with accuracy sufficient for practical purposes by the one-dimensional nonlinear heat equation

$$C(T)\frac{\partial T}{\partial \tau}=\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right),\quad T=T(x,\tau),\quad x\in(0,b),\quad \tau\in(\tau_{\min},\ \tau_{\max}]. \tag{1}$$

As boundary conditions for (1) we specify the initial temperature distribution

$$T(x,0)=T_0(x),\quad x\in[0,b] \tag{2}$$

and boundary conditions of the second kind

$$-\lambda(T)\frac{\partial T}{\partial x}(0,\tau)=q(\tau),\quad \tau\in(\tau_{\min},\tau_{\max}], \tag{3}$$

$$\lambda(T)\frac{\partial T}{\partial x}(b,\tau)=u(\tau),\quad \tau\in(\tau_{\min},\tau_{\max}], \tag{4}$$

where $u(\tau)$ is an unknown function. We assume that we have data on temperature measurements for the inner surface of the sample:

$$T_{\exp}(0,\tau)=f(\tau). \tag{5}$$

One of the methods for determining the unknown boundary condition $u(\tau)$ for a nonlinear heat equation is to solve the inverse problem by means of minimization of the root-mean-square dispersion of calculated temperature values at the points of fixing of thermal sensors $T_{\exp}$ from the experimentally measured values f. Two cases are possible: 1) we seek a solution in a finite dimensional space of parameters; 2) we solve the optimization problem in a functional space. The first approach is realized when the unknown function $u(\tau)$ is approximated by a certain system of basis functions, for example, by piecewise-constant functions [1], V-splines [2], etc.

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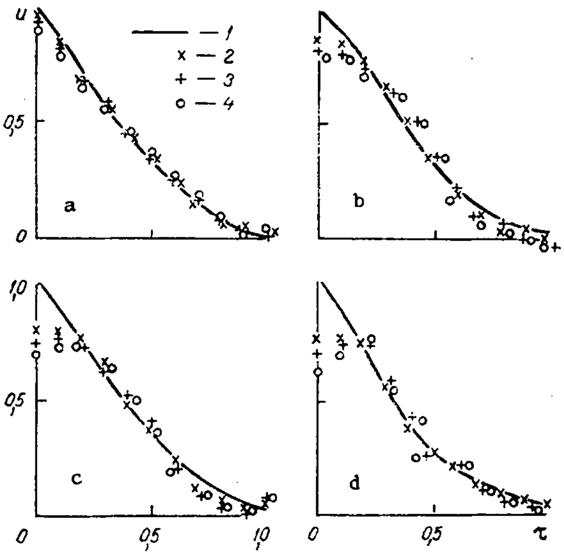

Fig. 1. Reconstruction of the dependence u(τ) [a) M = 3; b) 5; c) 7; d) 11]: 1) given dependences; 2) reconstructed values ($\delta_f = 0$); 3) $\delta_f = 0.05$; 4) 0.1.

It is shown in [2] with the help of a computational experiment that the approximation by cubic V-splines leads to an improvement in the convergence of the solution obtained to the given exact dependence as compared with the solution of the problem in a functional space. Therefore, the analysis of characteristics of the accuracy of the inverse heat conduction problem below will be conducted for this method of approximation.

Suppose we are given a segment D = $[\tau_{min}, \tau_{max}]$, where τ_{min} and τ_{max} are the starting time and completion time of the investigated process, respectively. We divide D into m equal parts and introduce a uniform mesh:

$$w = \{\tau_k = \tau_{min} + k\Delta\tau, \quad k = -2, \; -1, \; \ldots, \; m+3, \; \Delta\tau = (\tau_{max} - \tau_{min})/m\}.$$

The function

$$B^{(l-1)} = B^{(l-1)}(\tau_k, \; \tau_{k+1}, \; \ldots, \; \tau_{k+l}, \; \tau) = l \sum_{s=k}^{k+l} \frac{(\tau_s - \tau)_+^{l-1}}{w_k'}, \tag{6}$$

where $w_k' = (\tau - \tau_k)(\tau - \tau_{k+1}) \ldots (\tau - \tau_{k+\ell})$; $(\cdot)_+^{\ell-1} = \max\{0, (\cdot)^{\ell-1}\}$, is called a V-spline of $(\ell-1)$-th degree with respect to nodes $\tau_k, \tau_{k+1}, \ldots, \tau_{k+\ell}$ [3].

In order to realize algorithms practically we usually use splines of order not higher than three. In addition, in solving inverse problems we usually use V-splines with so-called "natural" boundary conditions:

$$u''(\tau_{min}) = u''(\tau_{max}) = 0.$$

The representation of the function sought by cubic V-splines on a uniform mesh is of the form

$$u(\tau) = \sum_{k=1}^{M} u_k \varphi_k(\tau),$$

$$\begin{gathered}
\varphi_1(\tau) = 2B_0(\bar{\tau} + \Delta\tau) + B_0(\bar{\tau}), \\
\varphi_2(\tau) = -B_0(\bar{\tau} + \Delta\tau) + B_0(\bar{\tau}), \\
\cdot \quad \cdot \quad \cdot \quad \cdot \\
\varphi_k(\tau) = B_{k-1}(\bar{\tau}), \quad k = \overline{3, \; M-2}, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\varphi_{M-1}(\tau) = B_0(\bar{\tau} - (M-3)\Delta\tau) - B_0(\bar{\tau} - (M-1)\Delta\tau),
\end{gathered} \tag{7}$$

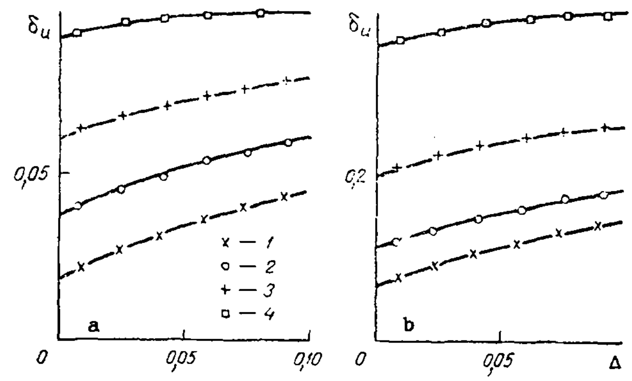

Fig. 2. Dependence of the accuracy of the solution on the accuracy of the initial data [a) in space L_2; b) in space C^0]: 1) M = 3; 2) 5; 3) 7; 4) 11.

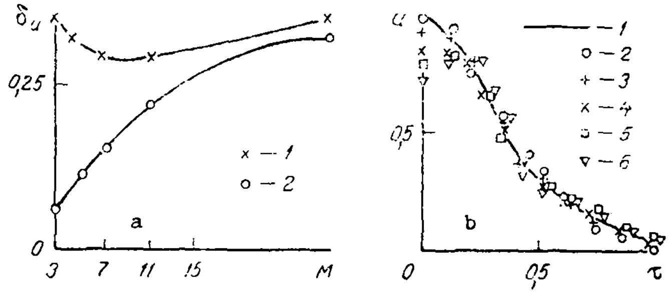

Fig. 3. Effect of the error of the approximation $\delta_A(M)$ on the error in the solution of the inverse heat conduction problem δ_u: a) the dependence of δ_u on M: 1) in the space L_2; 2) in the space C^o; b) results of the solution of the inverse heat conduction problem: 1) given values; 2) reconstructed values (M = 3); 3) M = 5; 4) 7; 5) 11; 6) 31.

$$\varphi_M(\tau) = B_0(\bar{\tau} - (M-2)\Delta\tau) + 2B_0(\bar{\tau} - (M-1)\Delta\tau), \tag{7}$$

where

$$\bar{\tau} = \tau - \tau_{\min};$$

$$B_k(\bar{\tau}) = B_0(\bar{\tau} - k\Delta\tau);$$

$$B_0(\bar{\tau}) = \frac{1}{6\Delta\tau^3}((\bar{\tau} + 2\Delta\tau)^3_+ - 4(\bar{\tau} + \Delta\tau)^3_+ + 6(\bar{\tau})^3_+ - 4(\bar{\tau} + \Delta\tau)^3_+ +$$

$$+ (\bar{\tau} - 2\Delta\tau)^3_+); \quad M = m + 1.$$

The function $B_0(\bar{\tau})$ has the property

$$B_0(\bar{\tau}) = \begin{cases} >0, & -2\Delta\tau < \bar{\tau} < 2\Delta\tau \\ =0, & |\bar{\tau}| \geqslant 2\Delta\tau. \end{cases}$$

Taking account of this property simplifies considerably the numerical realization of the solution of the inverse heat conduction problem.

We do not narrow the problem by letting $q(\tau) = 0$. The thickness b in the examples considered is assumed to be equal to 1. Thermophysical characteristics of the material were assumed to be constant $\lambda(T) = 1$, $C(T) = 1$.

The computational experiment has shown a connection between the conditioning of the algorithm of the solution of the inverse heat conduction problem and the number of unknown parameters approximating M. In Fig. 1, the results are shown obtained in the reconstruction

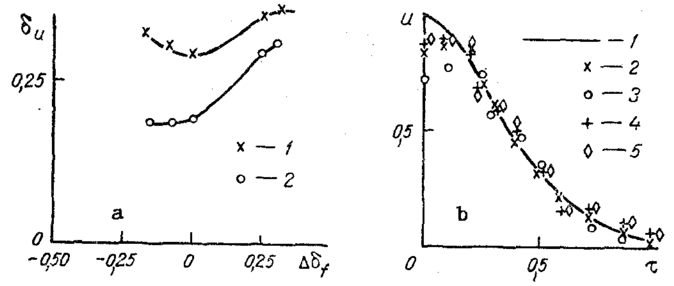

Fig. 4. Effect of the discrepancy error $\Delta\delta_f$ on the error in the solution of the inverse heat conduction problem δ_u: a) error in δ_u: 1) in the space L_2; 2) in space C^0; b) results of the solution of the inverse heat conduction problem: 1) given values; 2) reconstructed ($\Delta\delta_f = 0$); 3) $\Delta\delta_f = 0.25$; 4) −0.1; 5) −0.15.

of dependences $u(\tau)$ approximated exactly by V-splines for M = 3, 5, 7, 11 (Fig. 1a, b, c, and d, respectively). The effect of the inaccuracy of measurements due to thermal sensors was studied, calculated from

$$\tilde{f}(\tau) = f(\tau)(1 + \Delta\omega), \tag{8}$$

where $\Delta \in [0; 0.25]$; ω is a random variable distributed normally with zero mathematical expectation and unit dispersion, on the accuracy of the solution of the inverse heat conduction problem, calculated as follows:

$$\delta_u = \frac{\|\tilde{u} - u\|_{L_2}}{\|u\|_{L_2}} \tag{9}$$

and

$$\delta_u = \frac{\|\tilde{u} - u\|_{C^0}}{\|u\|_{C^0}}. \tag{10}$$

The results obtained (Fig. 2) exhibit a sharp impairment in the accuracy of the solution of the inverse heat conduction problem when the number of approximating parameters increases (especially in C metrics) in spite of the fact that the dependences sought are approximated exactly by corresponding V-splines.

Based on the calculations performed, we can conclude whether it is advisable to use for the approximation of the minimal number of nodes M of the V-spline the algorithm which allows us to achieve the condition for completing the iterative process [1]

$$J \leqslant \delta_f, \tag{11}$$

where J is the value of the minimizing functional; δ_f is the error in the temperature measurement metrically consistent with J.

This result is supported by the results obtained by varying the number of unknown parameters M (Fig. 3). The reconstructed dependence is approximated exactly by the V-spline for M = 7. Figure 3a shows the minimum of the error for the solution δ_u in the space L_2 for M = 7.

Another factor which affects the accuracy of the solution of the inverse problem is the error in specifying input data $\Delta\delta_f$. The value of error δ_f according to (11) determines the iteration number for which it is advisable to complete the iteration process [1]. Evidently, an overdetermination or underdetermination of the value of δ_f results in a certain violation of the principle of iterational regularization: An algorithm based on conjugate gradients might cease to be regularizing.

Figure 4 shows the corresponding results of the mathematical simulation. The error in specifying input data has been calculated as

$$\Delta\delta_f = \frac{\delta_f - \tilde{\delta}_f}{\delta_f}, \quad (12)$$

where δ_f is the exact value of the error; $\tilde{\delta}_f$ is the value known with an error. The results demonstrated show that the overdetermination of the level of error results in more appreciable errors in the solution of the inverse heat conduction problem in the metrics C^0 in comparison with the underdetermination. In comparing the accuracy δ_u in the metrics L_2, overdetermination and underdetermination in δ_f results in approximately equal results.

NOTATION

T, temperature; C, volumetric heat capacity; λ, thermal conductivity; f, additional temperature measurement; q, heat flux density to the internal body surface; u, heat flux density to the external surface; δ_f, measurement error; τ_{max}, duration of process; b, body thickness.

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DEGREE OF INSTABILITY OF NUMERICAL SOLUTIONS OF INVERSE HEAT-CONDUCTION PROBLEMS AND ERROR OF EXPERIMENTAL DATA

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UDC 536.6

A method is proposed for estimating the error of the results obtained in analyzing experimental data using the solutions of nonsteady boundary inverse heat-conduction problems.

An important aspect of the applied use of the solution of inverse heat-conduction problems is the question of determining the error of the results obtained [1-3]. In the present work, a solution of this problem is proposed for a sufficiently broad class of nonsteady boundary inverse heat-conduction problems in a linear formulation, expressed as an integral equation

$$\int_0^\tau q(t)\, G(\tau - t)\, dt = T_\delta(\tau) - T_0, \quad (1)$$

where $T_\delta(\tau)$ is the temperature dependence, measured with an error of δT; T_0 is the initial temperature; $G(\tau) = (\partial/\partial\tau)[U(\tau)]$.

Solution of Eq. (1) by approximating the desired heat flux as a piecewise-constant function (direct algebraic method [1]) is expressed by the following recurrence relation

$$q_i = \frac{1}{G_1}\left(T_i - T_0 - \sum_{j=1}^{i-1} q_j G_{i-j+1}\right), \quad i = 1, 2, \ldots, m. \quad (2)$$

Here $T_i = T_\delta(\tau_i)$, $\tau_i = i\Delta\tau$; q_i is the heat flux in the i-th time interval (τ_{i-1}, τ_i).

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